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Study of a Special Fractional Definite Integral

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***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we find the exact solution of a special fractional definite integral by using some methods. In fact, our result is a generalization of ordinary calculus result.**

***Keyword*: Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional definite integral.**

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-18]. There is no doubt that fractional calculus has become an exciting new mathematical method to solve diverse problems in mathematics, science, and engineering.

However, the rules of fractional derivative are not unique. Many authors have given the definition of fractional derivative. The commonly used definition is the Riemann-Liouvellie (R-L) definition. Other useful definitions include Caputo definition of fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [19-23]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we solve the following α-fractional definite integral:

$$\left({}_{0}I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}(1+x^{\alpha})\otimes_{\alpha}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right]^{\otimes_{\alpha}(-1)}\right],$$

where $0<\alpha\leq 1$, $\Gamma(\ \ )$ is the gamma function. The exact solution of this fractional definite integrals can be obtained by using some techniques. In addition, our result is a generalization of classical calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([24])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([25])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([26])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([27])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([28])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!} (f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([29])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([30])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Definition 2.8 ([31])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha m} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the mth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha (-1)}$.

Definition 2.9 ([32])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

Theorem 2.10 ([33])**:** *If* $0 < \alpha \leq 1$ *, then the* α*-fractional definite integral*

$$\left({}_0 I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha(1 + tan_\alpha(x^\alpha))\right] = \frac{T_\alpha}{16} \cdot Ln_\alpha(2). \tag{17}$$

III. MAIN RESULT

In this section, we solve a special fractional definite integral.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *then*

$$\left({}_0 I^\alpha_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha(1 + x^\alpha) \otimes_\alpha \left[1 + \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha (-1)}\right] = \frac{T_\alpha}{16} \cdot Ln_\alpha(2) . \tag{18}$$

Proof: Let $\frac{1}{\Gamma(\alpha+1)} x^\alpha = tan_\alpha(t^\alpha)$, then

$$\left({}_0 I^\alpha_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha(1 + x^\alpha) \otimes_\alpha \left[1 + \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha (-1)}\right]$$

$$= \left({}_{0}I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}} \right) \left[Ln_{\alpha}(1+x^{\alpha}) \otimes_{\alpha} \left[1 + \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_{\alpha} 2} \right]^{\otimes_{\alpha}(-1)} \otimes_{\alpha} \left({}_{0}D^{\alpha}_{x} \right) \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right] \right]$$

$$= \left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1) \cdot \frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}} \right) \left[Ln_{\alpha}(1+tan_{\alpha}(t^{\alpha})) \otimes_{\alpha} \left[1 + \left(tan_{\alpha}(t^{\alpha}) \right)^{\otimes_{\alpha} 2} \right]^{\otimes_{\alpha}(-1)} \otimes_{\alpha} \left({}_{0}D^{\alpha}_{t} \right) [tan_{\alpha}(t^{\alpha})] \right]$$

$$= \left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1) \cdot \frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}} \right) \left[Ln_{\alpha}(1+tan_{\alpha}(t^{\alpha})) \otimes_{\alpha} \left[\left(sec_{\alpha}(t^{\alpha}) \right)^{\otimes_{\alpha} 2} \right]^{\otimes_{\alpha}(-1)} \otimes_{\alpha} \left[\left(sec_{\alpha}(t^{\alpha}) \right)^{\otimes_{\alpha} 2} \right] \right]$$

$$= \left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1) \cdot \frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}} \right) \left[Ln_{\alpha}(1+tan_{\alpha}(t^{\alpha})) \right]$$

$$= \frac{T_{\alpha}}{16} \cdot Ln_{\alpha}(2) \ . \ \text{(by Theorem 2.10)} \qquad \text{Q.e.d.}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we solve a special fractional definite integral. In fact, our result is a generalization of traditional calculus result. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

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